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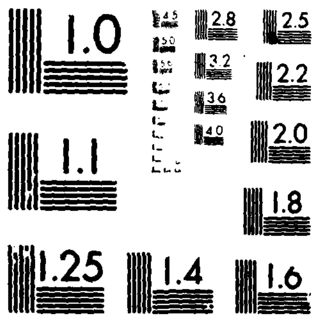

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(6) BOUNDS ON ORDER STATISTICS DIFFERENCES,

by

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Bounds of Order Statistics Differences

by

Salwa Fahmy and Frank Proschan

ABSTRACT

A sharp upper bound is obtained for the difference of a pair of order statistics in terms of the sample standard deviation.

1. Results and Derivation. In this note we obtain a sharp upper bound on the difference between a specified pair of sample order statistics in terms of the sample standard deviation. This result falls in the class of results discussed by Wolkowicz and Styan (1979), Arnold (1974), Mallows and Richter (1969), Samuelson (1968), and Thomson (1955).

Let $x_1 \le x_2 \le \ldots \le x_n$ denote the order statistics in a sample of size n. (The underlying distribution plays no role since we confine attention to the one sample only.) Let $\bar{x}$ denote the sample mean $n^{-1}\sum x_i$ and s^2 denote the sample variance $n^{-1}\sum(x_i - \bar{x})^2$.

Given integers k and ℓ $(1 \le k < \ell \le n)$, a sharp upper bound for the difference of order statistics is:

$$x_\ell - x_k \le s/a_{n,k,\ell}, \tag{1}$$

where

$$a^2_{n,k,\ell} = \frac{k(n - \ell + 1)}{n(n - \ell + 1 + k)} . \tag{2}$$

Equality holds in (1) if and only if:

$$x_1 = \ldots = x_k;\ x_{k+1} = \ldots = x_{\ell-1} = \frac{kx_k + (n - \ell + 1)x_\ell}{n - \ell + 1 + k};\ \text{and} \tag{3}$$

$$x_\ell = \ldots = x_n.$$

To prove inequality (1), fix x_k and x_ℓ $(1 \le k < \ell \le n)$ and consider s^2 as a function of the remaining order statistics. We search for the minimum value of s^2 in two stages. First it is clear that s^2 is minimized by requiring that $x_1 = \ldots = x_k$ and $x_\ell = \ldots = x_n$ irrespective of the values assumed by $x_{k+1}, \ldots, x_{\ell-1}$. Now consider s^2 as a function of $x_{k+1}, \ldots, x_{\ell-1}$. It can be shown through differentiation that the minimum value of s^2 is achieved when

$$x_i = \frac{kx_k + (n - \ell + 1)x_\ell}{n - \ell + 1 + k}, \; i = k + 1, \ldots, \ell - 1.$$

The minimum value of s^2 is computed to be

$$a^2_{n,k,\ell}(x_\ell - x_k)^2.$$

Hence (1) follows.

Some special cases of interest are:

Range: $x_n - x_1 \le (2n)^{\frac{1}{2}}s,$ (4)

Symmetric Case: $x_{n+1-k} - x_k \le (\frac{2n}{k})^{\frac{1}{2}}s,$ (5)

Spacing: $x_{k+1} - x_k \le [n/(k^{\frac{1}{2}}(n - k)^{\frac{1}{2}}]s.$ (6)

From (3), the corresponding sample values for which equality is achieved are readily computed.

Inequality (4) was obtained by Thomson (1955).

Remark. We have presented an upper bound on the difference $x_\ell - x_k$ of sample order statistics in terms of the sample standard deviation s (see (1)). Note that no comparable nontrivial lower bound exists for $x_\ell - x_k$, $(1 < k < \ell \le n$ or $1 \le k < \ell < n)$. This is apparent since s can be made arbitrarily large by choice of $x_1(x_n)$ in the first (second) case. However, a sharp lower bound exists for the sample range:

$$x_n - x_1 \ge 2s. \tag{7}$$

See Thomson (1955).

2. Applications. Consider the statistic $(x_\ell - x_k)/s$ which can be used to test for outliers. Inequality (1) shows that this statistic is bounded from above irrespective of the underlying distribution and gives the bound.

In life testing situations the approximate magnitude of s might be (statistically) known from previous experience. From this knowledge, inequality (6) would furnish a conservative estimate of the waiting time between the k^{th} and $k+1^{st}$ failures.

3. Numerical Examples. A few numerical examples may give the reader a clearer picture of the bounds in (1):

Symmetric Case: $n = 15$, $k = 3$: $x_{13} - x_3 \le 3.16s$

$n = 1000$, $k = 3$: $x_{998} - x_3 \le 25.82s$

Spacing: $n = 15$, $k = 9$: $x_{10} - x_9 \le 2.04s$

$n = 1000$, $k = 500$: $x_{501} - x_{500} \le 2.00s$.

Note that the bounds in the latter two cases are very close since the bound depends only on the ratio k/n.

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